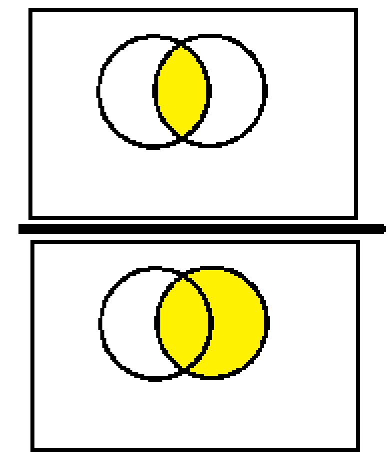
Stat 201: Introduction to Statistics

Standard 13: Probability – Conditional Probability

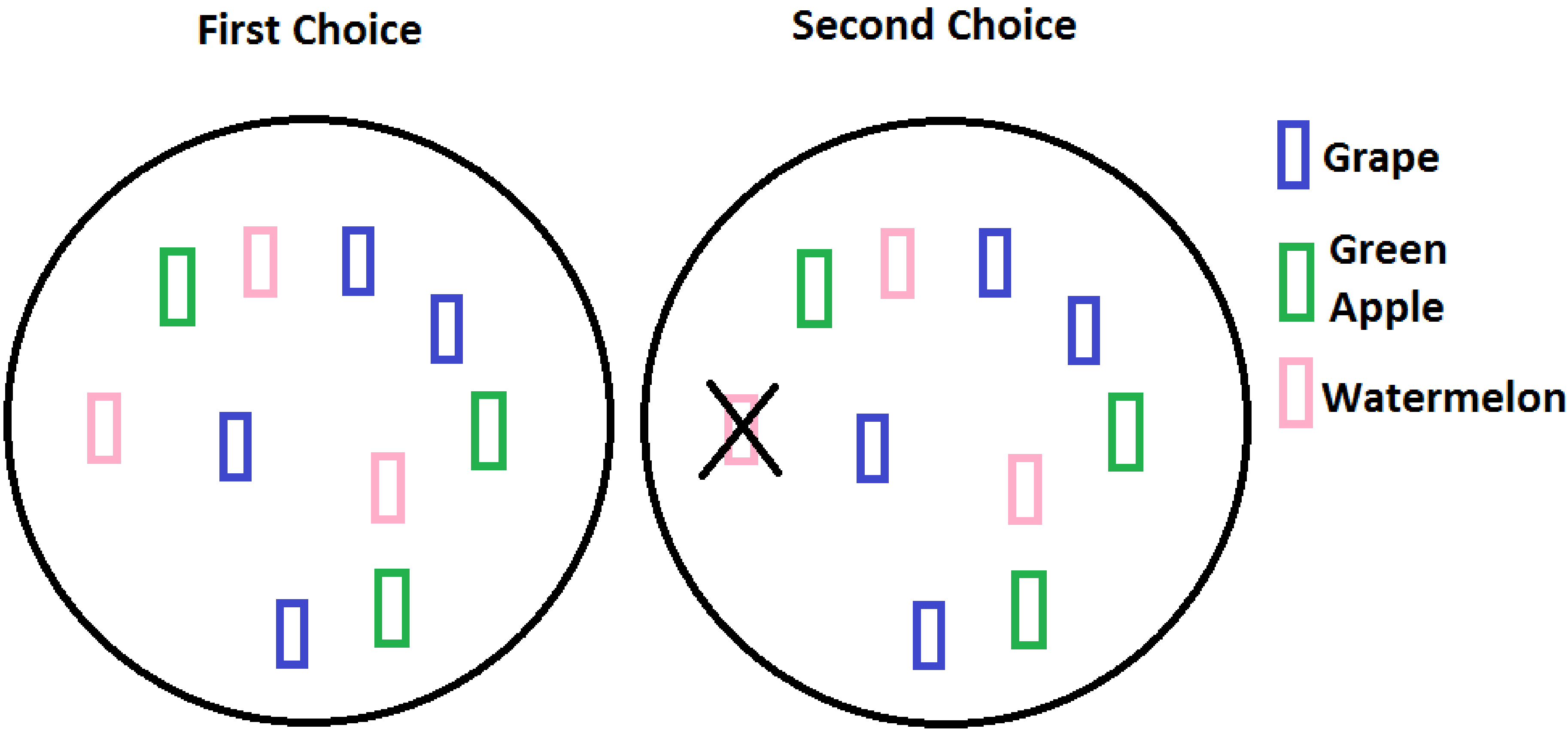
Probability Rules

- **Conditional Probabilities**
  - The probability of A given B:

$$P(A|B) = \frac{P(A\ and\ B)}{P(B)} = \frac{P(A \cap B)}{P(B)} =$$



Conditional Probabilities



Walkthrough

- Sampling with or without Replacement*
  - https://www.youtube.com/watch?v=uKTjh-6PFjo

- Conditional Probabilities*
  - https://www.youtube.com/watch?v=JGeTcRfKgBo

Example: Probability

↓Velociraptor safe windows?	Survived Velociraptor Attacks (S)	Devoured by Velociraptors (D)	Total
Yes (Y)	412,368	510	**412,878**
No (N)	16,001	162,527	178,528
Total	428,369	163,037	**591,406**

- The probability a randomly selected participant had Velociraptor safe windows:

$$\widehat{P(Y)} = \frac{number\ of\ Y\ observations}{total\ number\ of\ observations} = \frac{412{,}878}{591{,}406} = .69812954$$

Example: Probability

↓Velociraptor safe windows?	Survived Velociraptor Attacks (S)	Devoured by Velociraptors (D)	Total
Yes (Y)	**412,368**	510	412,878
No (N)	16,001	162,527	178,528
Total	428,369	163,037	**591,406**

- The probability a randomly selected participant survived the Velociraptor attacks **<u>and</u>** had Velociraptor safe windows:
- $P(\widehat{S \cap Y}) = \frac{number\ of\ S\&Y\ observations}{total\ number\ of\ observations}$

$$= \frac{412{,}368}{591{,}406} = .69726719$$

Example: Probability

↓Velociraptor safe windows?	Survived Velociraptor Attacks (S)	Devoured by Velociraptors (D)	Total
Yes (Y)	412,368	510	412,878
No (N)	16,001	162,527	178,528
Total	428,369	163,037	591,406

- The probability a randomly selected participant survived the Velociraptor attacks **given** they had Velociraptor safe windows
- **Let's try the formula:**
- $\widehat{P(S|Y)} = \frac{P(S\&Y)}{P(Y)} = \frac{.69726719}{.69812954} = .99876477$

Example: Probability

↓Velociraptor safe windows?	Survived Velociraptor Attacks (S)	Devoured by Velociraptors (D)	Total
Yes (Y)	**412,368**	510	**412,878**
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- The probability a randomly selected participant survived the Velociraptor attacks **given** they had Velociraptor safe windows:
- $\widehat{P(S|Y)} = \frac{number\ of\ S\&Y\ observations}{total\ number\ of\ Y\ observations} = \frac{412{,}368}{412{,}878} = .99876477$

Example: Probability

- The probability a randomly selected participant survived the Velociraptor attacks given they had Velociraptor safe windows:

<u>*Proof:*</u>

- $P(S|Y) = \frac{P(S \cap Y)}{P(Y)} = \frac{\left(\frac{number\ of\ S\&Y\ observations}{total\ number\ of\ observations}\right)}{\left(\frac{number\ of\ Y\ observations}{total\ number\ of\ observations}\right)}$

$$= \left(\frac{number\ of\ S\&Y\ observations}{\cancel{total\ number\ of\ observations}}\right)\left(\frac{\cancel{total\ number\ of\ observations}}{number\ of\ Y\ observations}\right)$$

$$= \frac{number\ of\ S\&Y\ observations}{total\ number\ of\ Y\ observations}$$

Example: Probability

↓Velociraptor safe windows?	Survived Velociraptor Attacks (S)	Devoured by Velociraptors (D)	Total
Yes (Y)	412,368	510	412,878
No (N)	16,001	162,527	178,528
Total	428,369	163,037	591,406

- The probability a randomly selected participant survived the Velociraptor attacks **and** had Velociraptor safe windows:
- **Let's try the formula:**
- $P(\widehat{Y \cap S}) = P(Y) * P(S|Y)$

$$= .69812954 * .998764777$$
$$= .69726719$$

Example: Probability

↓Velociraptor safe windows?	Survived Velociraptor Attacks (S)	Devoured by Velociraptors (D)	Total
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- Now, let’s think about a word problem.
- Let’s say Velociraptors came back, maybe there was a cryogenically frozen one that thawed out. Anyway, **would it be worth getting the Velociraptor safe windows?**

Example: Probability

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Yes (Y)	412,368	510	412,878
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- At first look…
  - We see that a lot more people survived the Velociraptor attacks with the Velociraptor safe windows
  - We see that a lot more were devoured without the Velociraptor safe windows
- **But** counts can be misleading because there are far more people with the windows than without

Example: Probability

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Yes (Y)	412,368	510	412,878
No (N)	16,001	162,527	178,528
Total	428,369	163,037	591,406

- We know…
  - The probability we survive **<u>given</u>** we have Velociraptor save windows is

$$P(S|Y) = .99876477$$

Example: Probability

↓Velociraptor safe windows?	Survived Velociraptor Attacks (S)	Devoured by Velociraptors (D)	Total
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Total	428,369	163,037	591,406

- We can figure out…
  - The probability we survive **given** we don't have Velociraptor save windows is

$$P(S|N) = \frac{P(S\&N)}{P(N)} = \frac{\left(\frac{16{,}001}{591{,}406}\right)}{\left(\frac{178{,}528}{591{,}406}\right)} = \ .089627397$$

Example: Probability

↓Velociraptor safe windows?	Survived Velociraptor Attacks (S)	Devoured by Velociraptors (D)	Total
Yes (Y)	412,368	510	412,878
No (N)	16,001	162,527	178,528
Total	428,369	163,037	591,406

- So now we know…
  - The probability we survive **given** we have Velociraptor save windows is

    $$P(S|Y) = .99876477$$

  - The probability we survive **given** we don't have Velociraptor save windows is

    $$P(S|N) = .089627397$$

Example: Probability

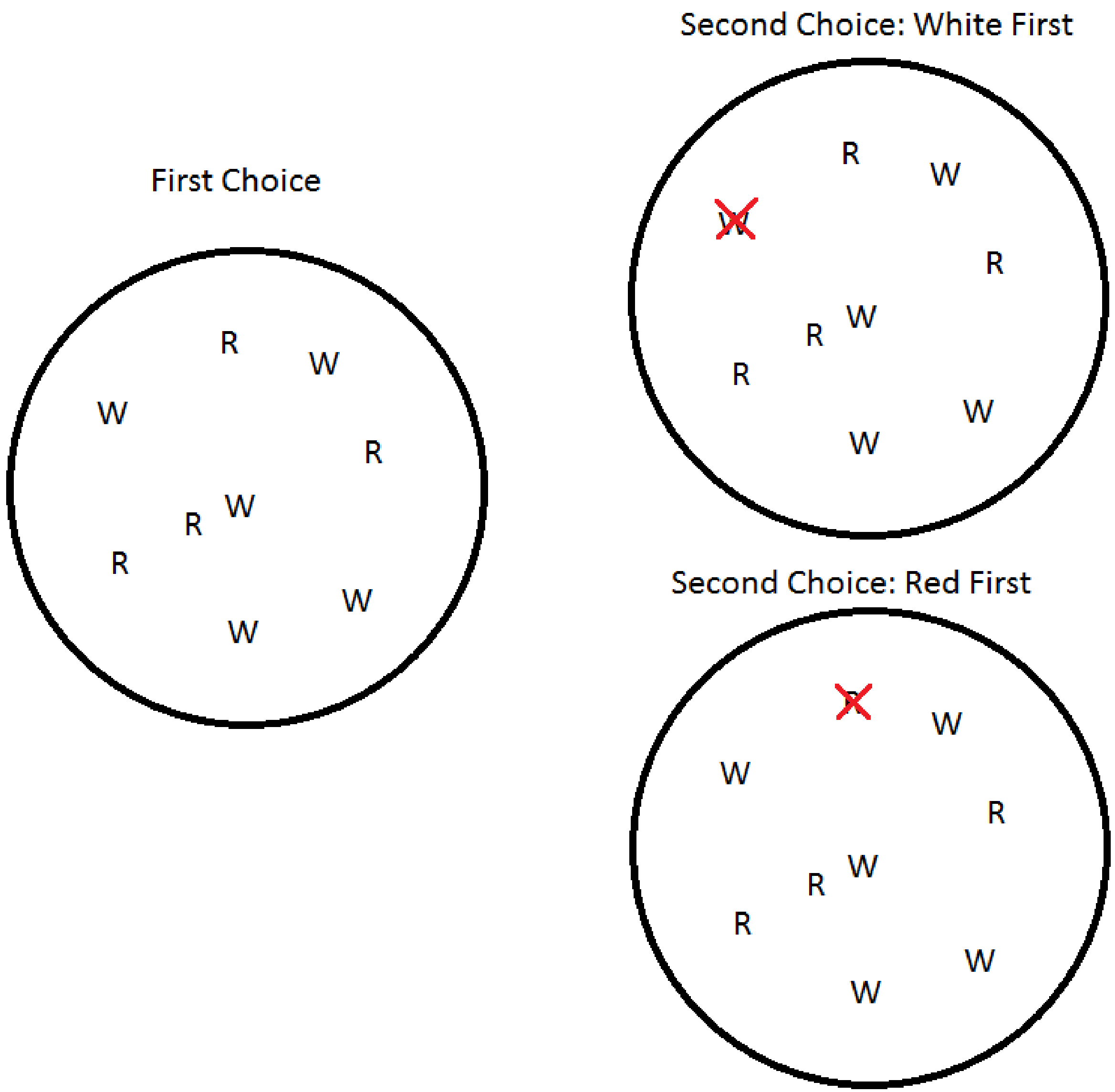
↓Velociraptor safe windows?	Survived Velociraptor Attacks (S)	Devoured by Velociraptors (D)	Total
Yes (Y)	412,368	510	412,878
No (N)	16,001	162,527	178,528
Total	428,369	163,037	591,406

- From what we know…
  - $P(S|Y) = \ .99876477$
  - $P(S|N) = .089627397$
  - It is much more likely that we would survive if we bought the velociraptor safe windows – we should definitely invest in some!

Example 2: Flowers

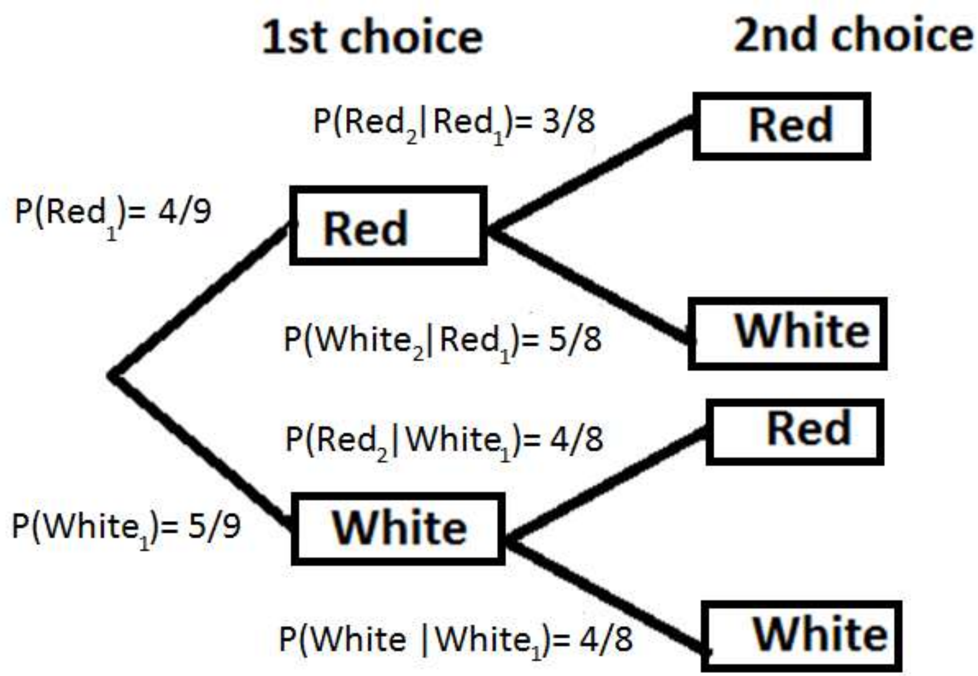
- 9 seeds: 4 are red and 5 are white
- Choose 2 seeds at random:
  - One at a time **without replacement**
  - Without replacement means exactly what it sounds like – we don't put our first choice back
  - In other words, think of it this way – you choose one seed, plant it and it's gone forever, and then you choose another from the sack

Example 2: Flowers



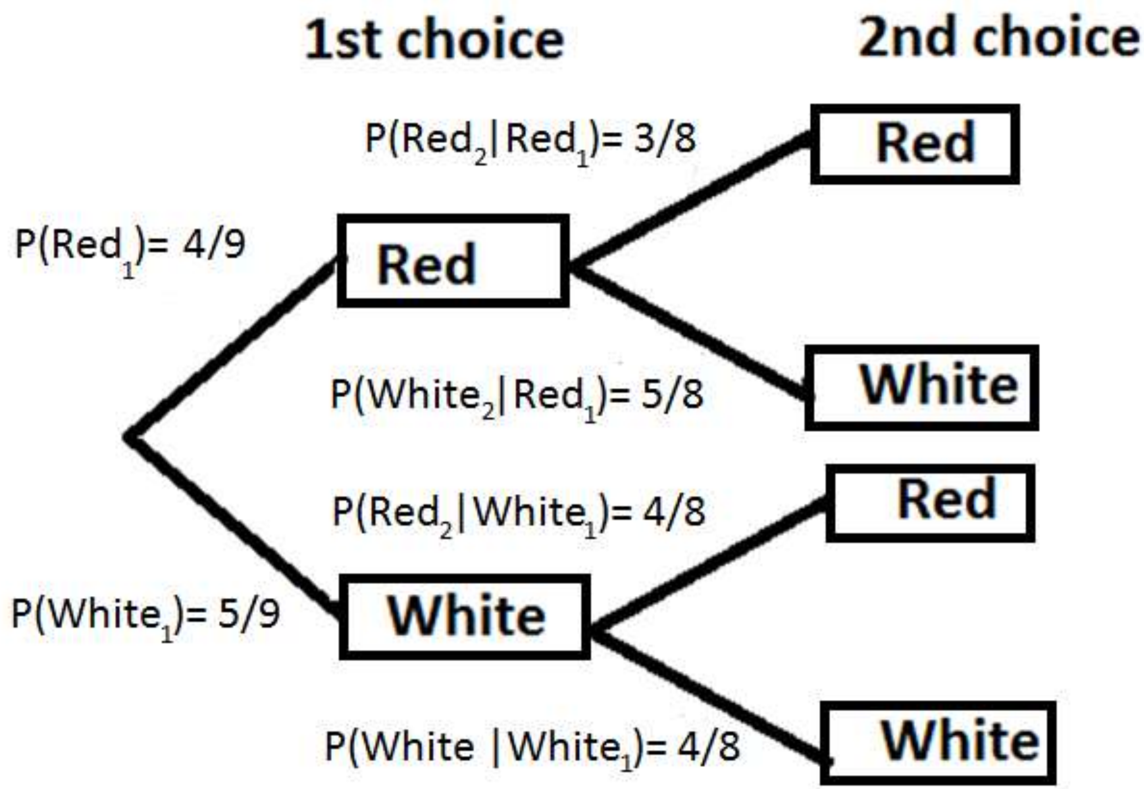
Example 2: Flowers

- The first choice is from nine seeds
  - 4 are red
  - 5 are white



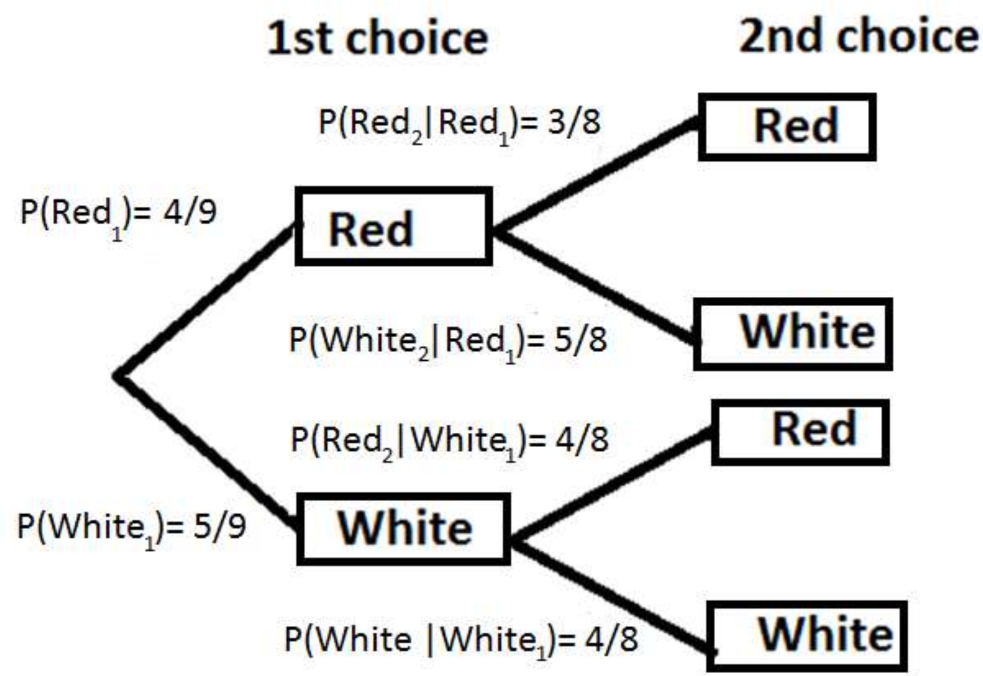
Example 2: Flowers

- The second choice is from eight seeds because we chose one without replacement



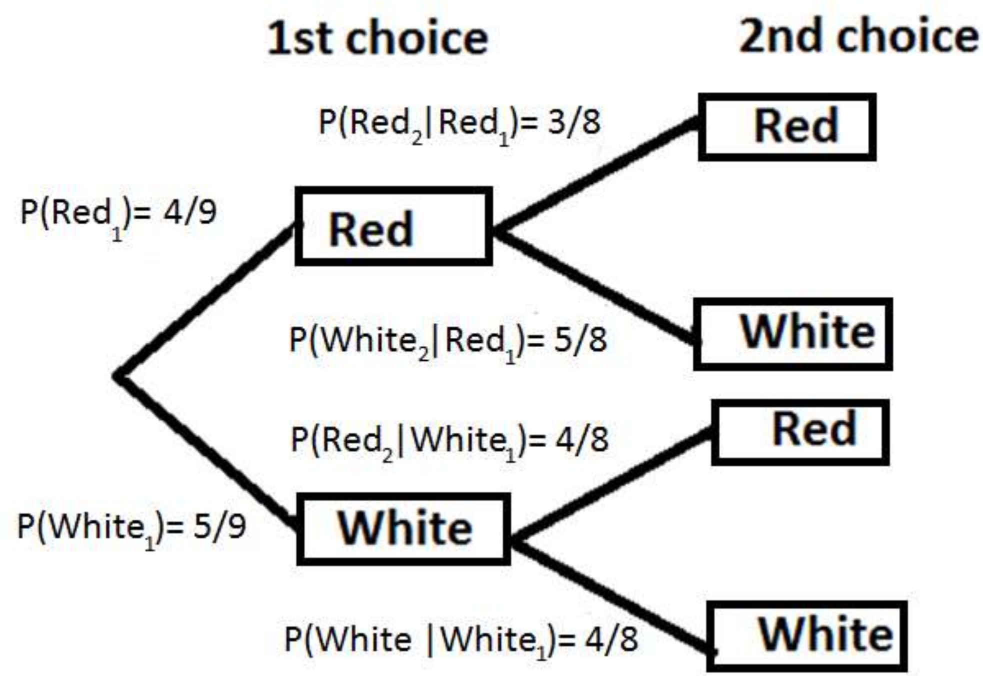
Example 2: Flowers

- The second choice is from eight seeds
- If the first was red:
  - 3 are red
  - 5 are white



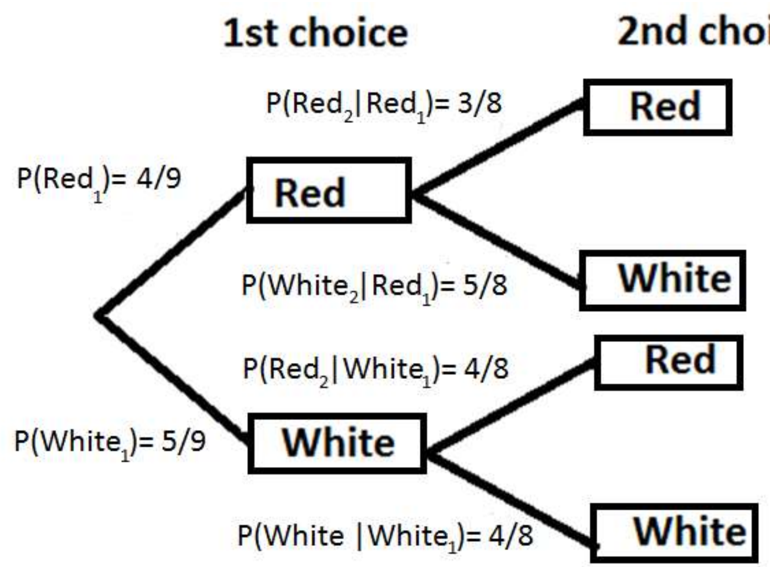
Example 2: Flowers

- The second choice is from eight seeds
- If the first was white:
  - 4 are red
  - 4 are white



Example 2: Flowers

- 9 seeds: 4 are red and 5 are white
- Choose 2 seeds at random without replacement

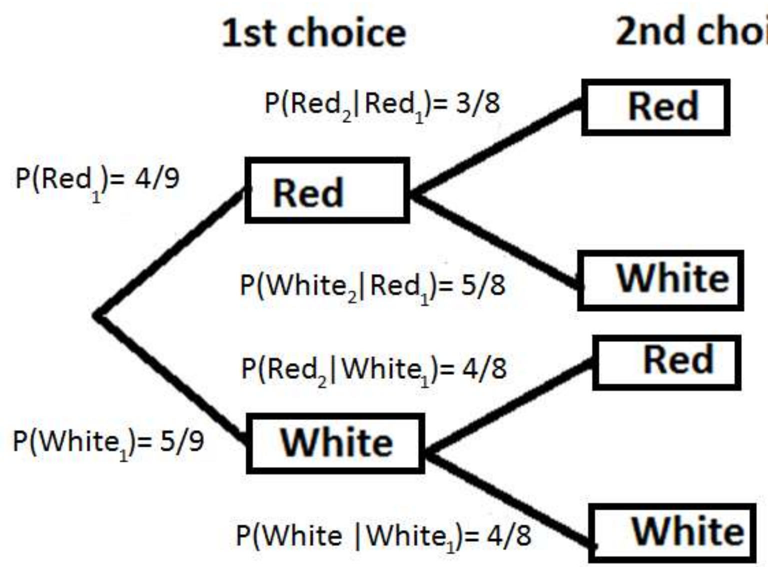


- The probability of selecting a red on our first try

$$P(Red_1) = \frac{4}{9}$$

Example 2: Flowers

- 9 seeds: 4 are red and 5 are white
- Choose 2 seeds at random without replacement



- The probability of selecting a white on our first try
- $P(White_1) = \frac{5}{9}$

Example 2: Flowers

- The probability of selecting a red on our second try given we got a red on our first try
  - We started with nine seeds and we selected one without replacement, so now we have eight seeds
  - We started with four red seeds and we selected a red on our first try, so now we have three red seeds

$$P(Red_2|Red_1) = \frac{3}{8}$$

Example 2: Flowers

- The probability of selecting a red on our second try given we got a red on our first try
  - We started with nine seeds and we selected one without replacement, so now we have eight seeds
  - We started with five white seeds and we selected a red on our first try, so we still have five white seeds

$$P(White_2|Red_1) = \frac{5}{8}$$

Example 2: Flowers

- The probability of selecting a red on our second try given we got a red on our first try
  - We started with nine seeds and we selected one without replacement, so now we have eight seeds
  - We started with four red seeds and we selected a white on our first try, so we still have four red seeds

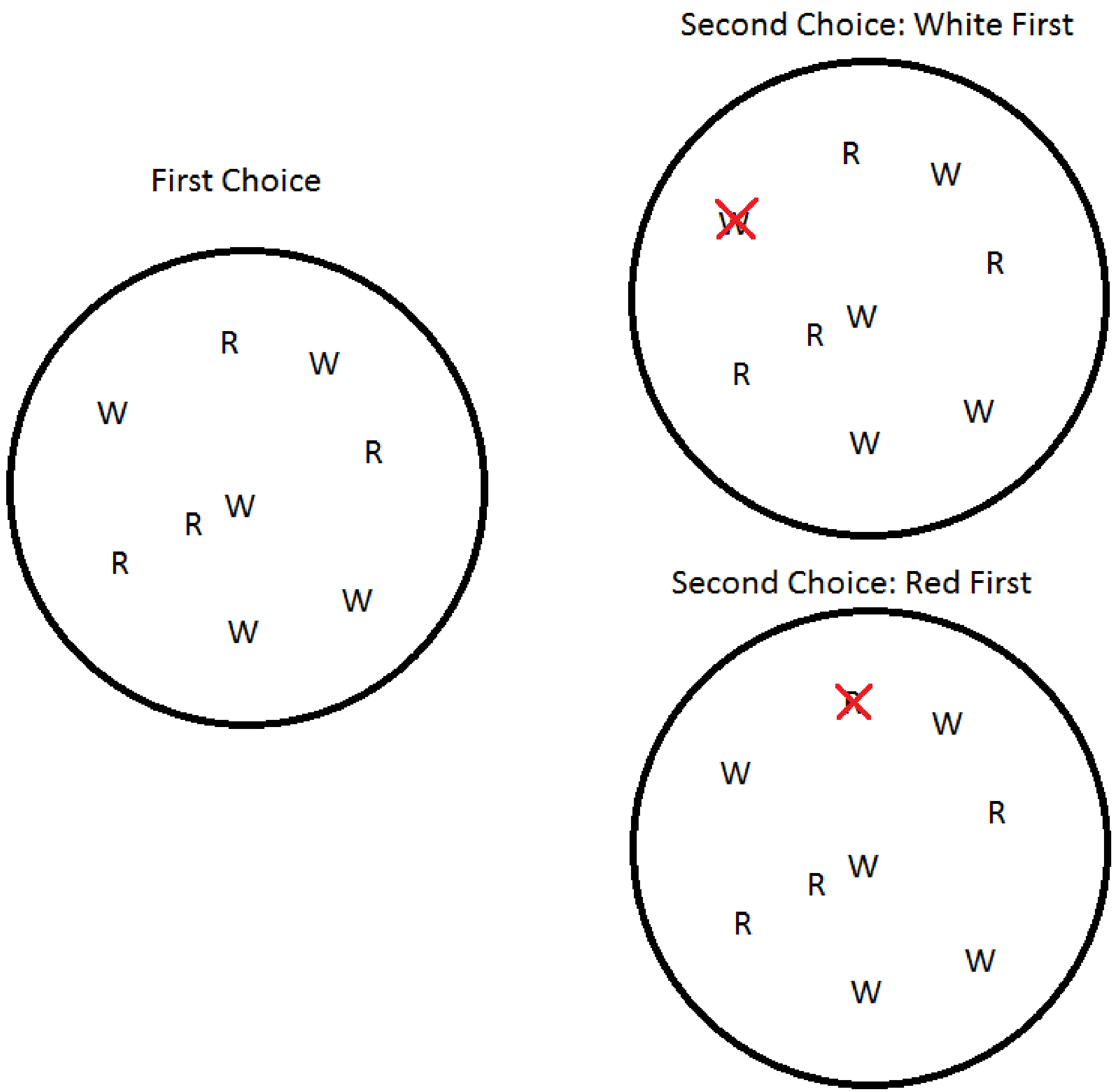
$$P(Red_2|White_1) = \frac{4}{8}$$

Example 2: Flowers

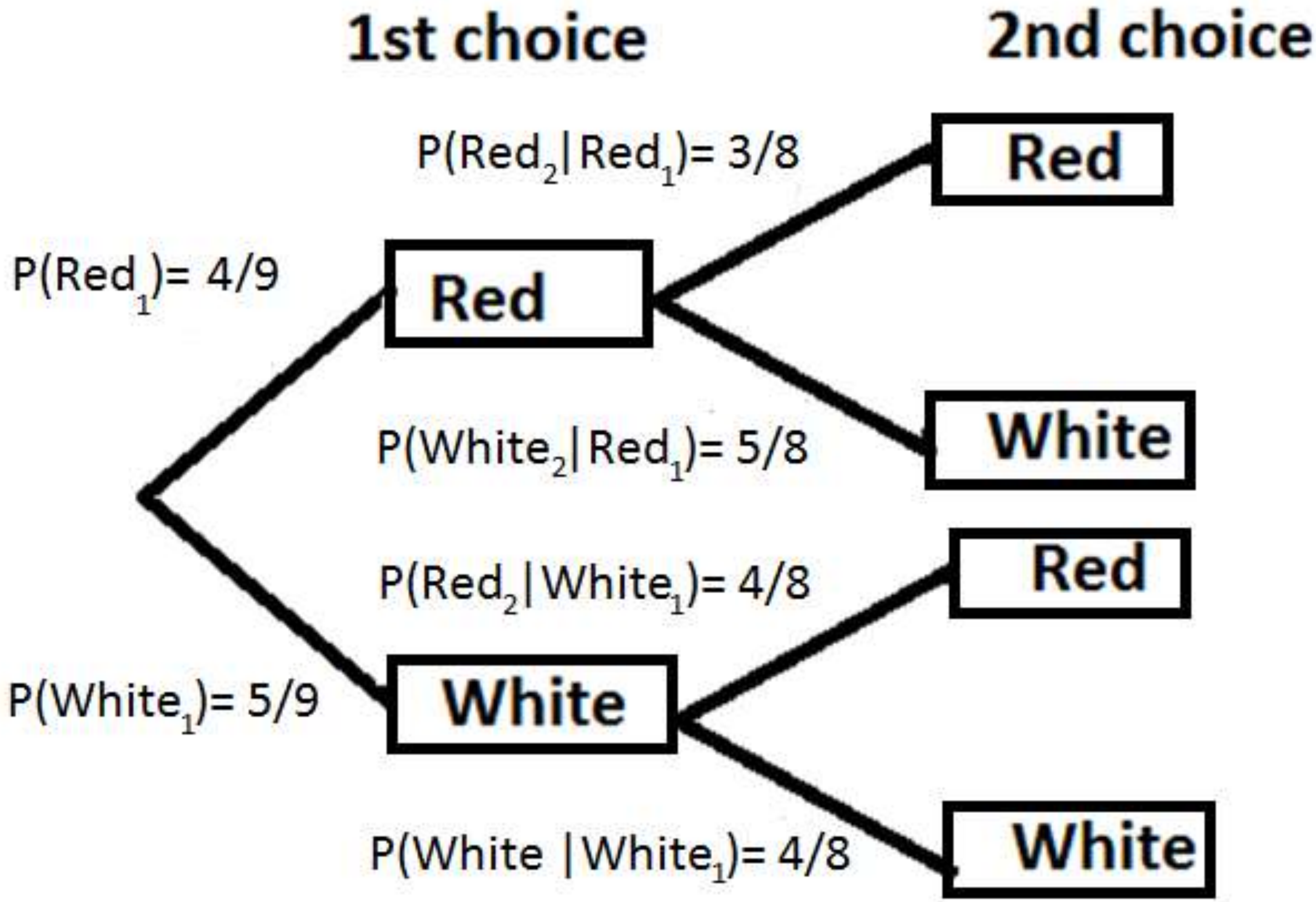
- The probability of selecting a red on our second try given we got a red on our first try
  - We started with nine seeds and we selected one without replacement, so now we have eight seeds
  - We started with five white seeds and we selected a white on our first try, so now we have four white seeds

$$P(White_2|White_1) = \frac{4}{8}$$

Example 2: Flowers



Example 2: Flowers



Example 2: Flowers

- To find ‘and’ probabilities we just multiply across the branches
- Remember P(A and B) = P(A) ∗ P(B|A)
- The probability of choosing two red seeds in a row – i.e. a red first **and** a red second

$$P\left(R_1 \bigcap R_2\right) = P(R_1) * P(R_2|R_1)$$

$$= \left(\frac{4}{9}\right) * \left(\frac{3}{8}\right) = \left(\frac{1}{6}\right)$$

Example 2: Flowers

- To find ‘and’ probabilities we just multiply across the branches
- Remember $\mathrm{P}(\mathrm{A} \cap B) = \mathrm{P}(\mathrm{A}) * \mathrm{P}(\mathrm{B}|\mathrm{A})$
- The probability of choosing a red seed first, **and** then a white seed second

$$P\left(R_1 \bigcap W_2\right) = P(R_1) * P(W_2|R_1)$$

$$= \left(\frac{4}{9}\right) * \left(\frac{5}{8}\right) = \left(\frac{5}{18}\right)$$

Example 2: Flowers

- To find ‘and’ probabilities we just multiply across the branches
- Remember $\mathrm{P}(\mathrm{A} \cap B) = \mathrm{P}(\mathrm{A}) * \mathrm{P}(\mathrm{B}|\mathrm{A})$
- The probability of choosing a white seed first, and then a red seed

$$P\left(W_1 \bigcap R_2\right) = P(W_1) * P(R_2|W_1)$$

$$= \left(\frac{5}{9}\right) * \left(\frac{4}{8}\right) = \left(\frac{5}{18}\right)$$

Example 2: Flowers

- To find ‘and’ probabilities we just multiply across the branches
- Remember $\mathrm{P}(\mathrm{A}\ \cap B) = \mathrm{P}(\mathrm{A}) * \mathrm{P}(\mathrm{B}|\mathrm{A})$
- The probability of choosing two white seeds in a row

$$P(W1\ and\ W2) = P(W_1) * P(W_2|W_1)$$

$$= \left(\frac{5}{9}\right) * \left(\frac{4}{8}\right) = \left(\frac{5}{18}\right)$$

Example 2: Flowers

- To find 'and' probabilities we just multiply across the branches
- Remember $\mathrm{P}(\mathrm{A} \cap B) = \mathrm{P}(\mathrm{A}) * \mathrm{P}(\mathrm{B}|\mathrm{A})$
- The probability of choosing a white and a red, regardless of order

$$P(1\, red\, \&\, 1\, white) = P(R_1 \& W_2\ or\ W_1 \& R_2)$$
$$= P(R_1 \& W_2) + P(W_1 \& R_2)$$
$$= \left(\frac{5}{18}\right) + \left(\frac{5}{18}\right) = \left(\frac{10}{18}\right) = \left(\frac{5}{9}\right)$$

Probability

Type	Description
Conditional Probability ('A given B')	$P(A\|B) = \frac{P(A \cap B)}{P(B)}$
And Probability ('A and B')	$\mathrm{P}\left(A \bigcap B\right) = \mathrm{P(A)} * \mathrm{P(B\|A)}$ $= \mathrm{P(B)} * \mathrm{P}(\mathrm{A}\|B)$